Concurrent Quantum Separation Logic for Fine-Grained Parallelism

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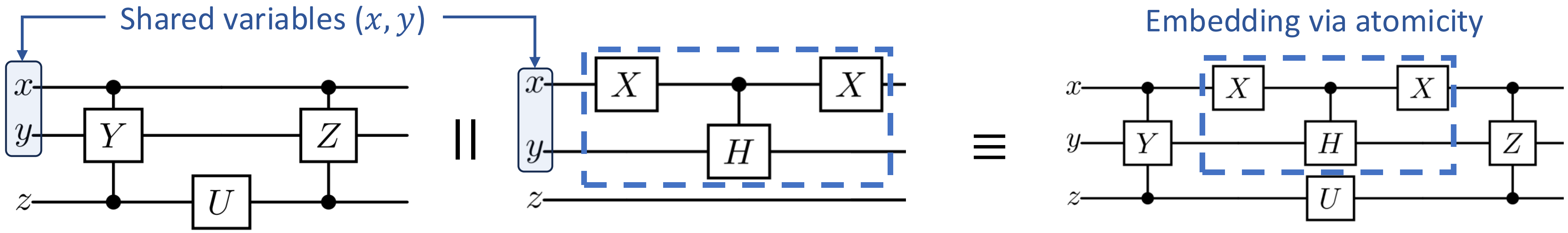
Overview of Our Work

We propose **concurrent quantum separation logic** for **modularly** verifying quantum programs with fine-grained parallelism

- Compared to existing quantum SLs [Zhou+ LICS'21] [Le+ POPL'22], our logic is the first to support concurrency and the sharing of quantum resources, and can verify non-trivial programs

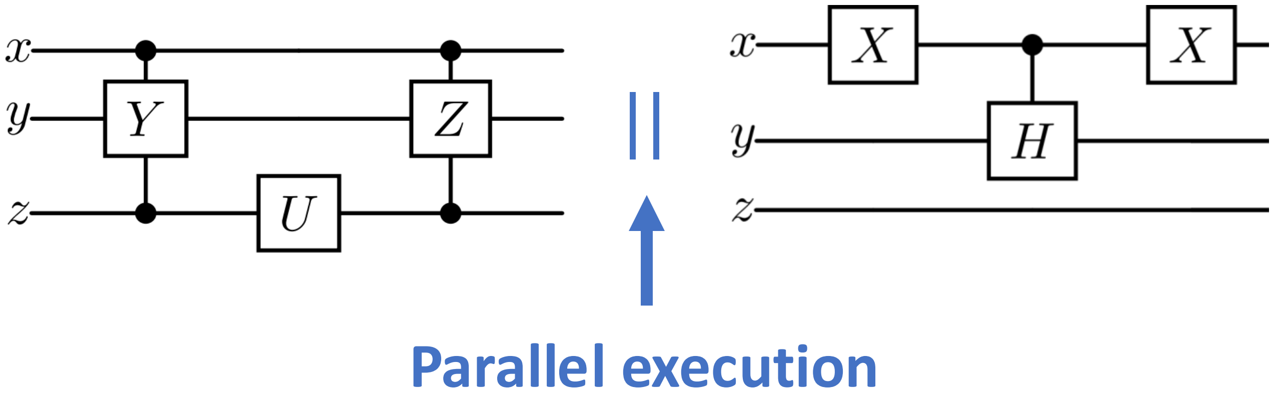


Outline

- **Motivation: Parallelizing Quantum Programs**
- Our Work: Concurrent QSL for Fine-Grained Parallelism
- Extension to Probabilistic Reasoning & Conclusion

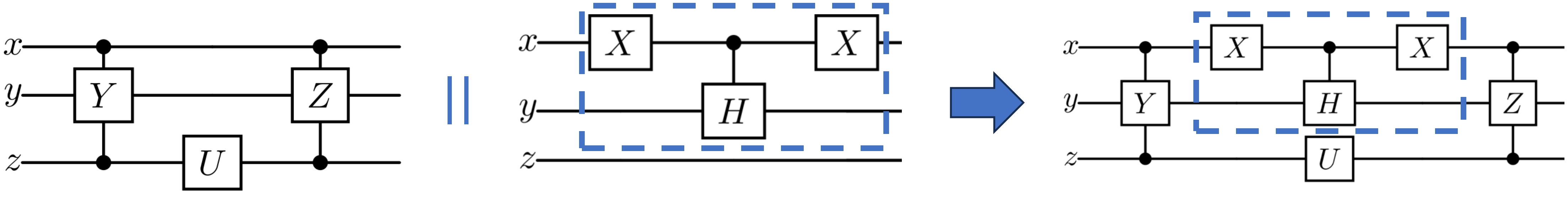
Parallelizing Quantum Programs

- Parallelizing quantum programs can reduce execution costs

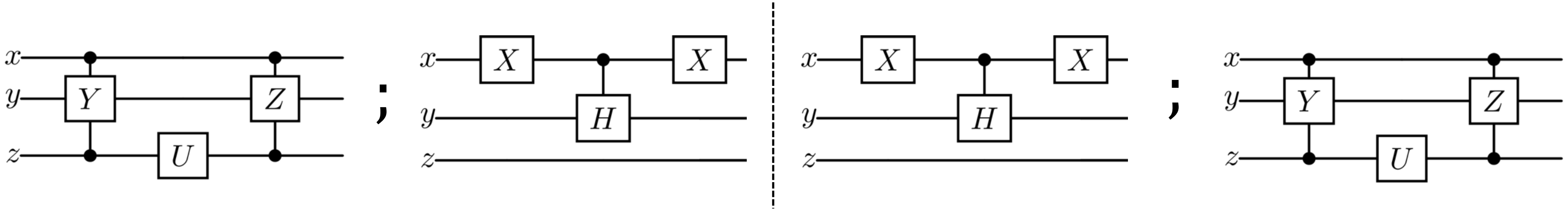


Parallelizing Quantum Programs

- Parallelizing quantum programs can reduce execution costs



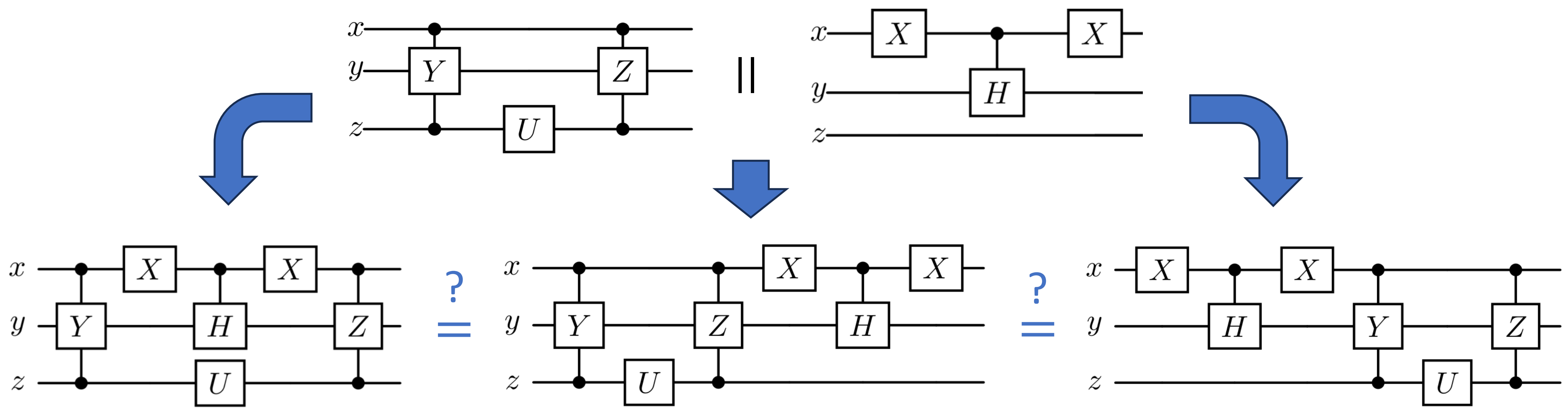
Patterns of parallelization results = Possible execution traces



Verifying the Correctness of Parallelization

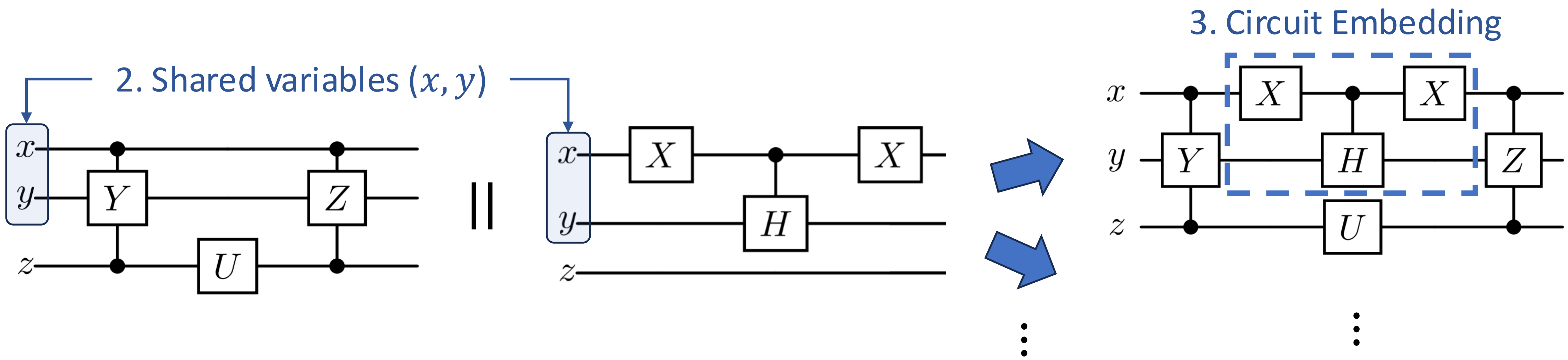
Parallelization is correct? = Concurrent program is correct?

Correctness of a parallel program ≈ **Uniqueness of the output**



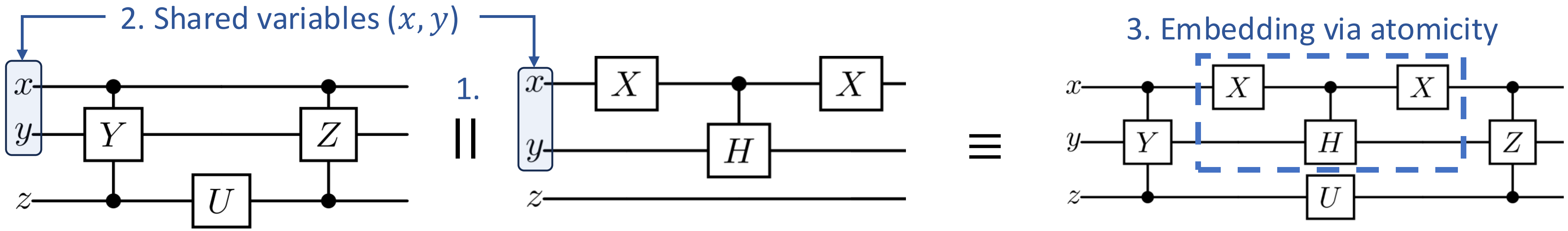
What is Challenge?

1. Parallelization allows exponentially many execution traces!
   ⇒ Need a **modular** method for parallel quantum programs
2. Concurrent programs may share qubits in non-trivial way
3. Support circuit embedding in non-interfered way



Our Work: Concurrent Quantum Separation Logic for Fine-Grained Parallelism

1. **Support parallel execution of quantum processes**
2. **Support shared quantum variables**
   - Even when there are apparent write-write races
3. **Support atomic expressions**
   - For non-interfered embedding of quantum circuits



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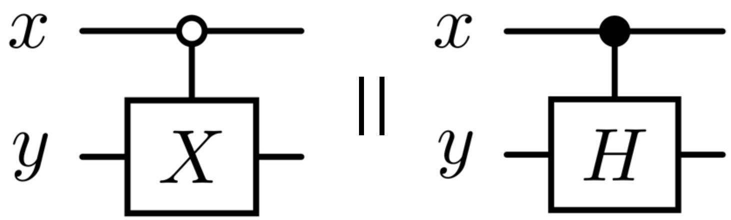
A Simple Example

$$C_0X(x,y)\ ||\ C_1H(x,y)$$



Our Goal: Prove this

Precondition $\{\ (x,y) \mapsto (\alpha|0\rangle|\phi_0\rangle + \beta|1\rangle|\phi_1\rangle)\ *\ [y]\ \}$

$$C_0X(x,y)\ ||\ C_1H(x,y)$$

Postcondition $\{\ (x,y) \mapsto (\alpha|0\rangle \otimes X|\phi_0\rangle + \beta|1\rangle \otimes H|\phi_1\rangle) * [y]\ \}$

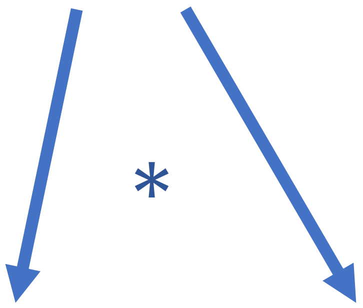
- ***Quantum points-to token*** $\bar{x} \mapsto |\psi\rangle$: the state vector of $\bar{x}$ is $|\psi\rangle$
- Separation $*$ means disentangled qubit states:

$$\bar{x} \mapsto |\psi\rangle\ *\ \bar{y} \mapsto |\phi\rangle\ \equiv\ (\bar{x},\bar{y}) \mapsto |\psi\rangle \otimes |\phi\rangle$$

- ***Qubit token*** $[x]$ (new!): Qubit x is alive, but its state is unknown

Proof for $C_0X(x,y) \mid\mid C_1H(x,y)$

$$\{\,(x,y) \mapsto (\alpha|0\rangle|\phi_0\rangle + \beta|1\rangle|\phi_1\rangle) * [y]\,\}$$



$$C_0X(x,y) \;\mid\mid\; C_1H(x,y)$$

Quantum resources (propositions) can be distributed to processes by separation *

$\Rightarrow$ How?

Remark:

- $y \mapsto |\psi\rangle * y \mapsto |\psi\rangle$ is not allowed
- x and y may not be separable

$$\{\,(x,y) \mapsto (\alpha|0\rangle \otimes X|\phi_0\rangle + \beta|1\rangle \otimes H|\phi_1\rangle) * [y]\,\}$$

Our Key Observation

$$C_0X(x, y) \mid\mid C_1H(x, y)$$

- Both processes can write to y simultaneously due to superposition
  - If $x \mapsto \alpha|0\rangle + \beta|1\rangle$ for $\alpha, \beta \neq 0$, then both C_0X and C_1H update y
- How to distribute "write permission" on y to both processes?
- **Our idea:** **Quantum case analysis over the bases** of a qubit x

$x \mapsto |0\rangle$

$C_0X(x, y) \mid\mid \underline{C_1H(x, y)}$

Write permission is not required

$x \mapsto |1\rangle$

$\underline{C_0X(x, y)} \mid\mid C_1H(x, y)$

- After the case analysis, only one process writes to the qubit
  $\Rightarrow$ The apparent write-write race is eliminated!

Linear Combination Rule

This idea can be formalized as **linear combination of Hoare triples**

$$\frac{\{\ \bar{x} \mapsto |\psi\rangle * P\ \}\ e\ \{\ \bar{x} \mapsto |\phi\rangle * Q\ \} \qquad \{\ \bar{x} \mapsto |\psi'\rangle * P\ \}\ e\ \{\ \bar{x} \mapsto |\phi'\rangle * Q\ \}}{\{\ \bar{x} \mapsto (\alpha|\psi\rangle + \beta|\psi'\rangle)\ * P\ \}\ e\ \{\ \bar{x} \mapsto (\alpha|\phi\rangle + \beta|\phi'\rangle)\ * Q\ \}}$$

Now Our Subgoals:

$$\{\ (x,y) \mapsto |0\rangle|\phi_0\rangle * [y]\ \}\ C_0X(x,y)\ ||\ C_1H(x,y)\ \{\ (x,y) \mapsto |0\rangle \otimes X|\phi_0\rangle * [y]\ \}$$

$$\{\ (x,y) \mapsto |1\rangle|\phi_1\rangle * [y]\ \}\ C_0X(x,y)\ ||\ C_1H(x,y)\ \{\ (x,y) \mapsto |1\rangle \otimes H|\phi_1\rangle * [y]\ \}$$

Proof for $C_0X(x,y) \mid\mid C_1H(x,y)$

$\{\ (x,y) \mapsto (\alpha|0\rangle|\phi_0\rangle + \beta|1\rangle|\phi_1\rangle) * [y]\ \}$

$\{\ x \mapsto |0\rangle * y \mapsto |\phi_0\rangle * [y]\ \}$

$\{\ x \mapsto |1\rangle * y \mapsto |\phi_1\rangle * [y]\ \}$

$y \mapsto |\phi_0\rangle$ $*$ $[y]$

$C_0X(x,y) \mid\mid C_1H(x,y)$

$\{\ x \mapsto |0\rangle * y \mapsto X|\phi_0\rangle * [y]\ \}$

$\{\ x \mapsto |1\rangle * y \mapsto H|\phi_1\rangle * [y]\ \}$

$\{\ (x,y) \mapsto (\alpha|0\rangle \otimes X|\phi_0\rangle + \beta|1\rangle \otimes H|\phi_1\rangle) * [y]\ \}$

- Give $y \mapsto |\phi_i\rangle$ to i-th process
- x is required by both processes as “read-only” qubits

Remark:

- $x \mapsto |\psi\rangle * x \mapsto |\psi\rangle$ is not allowed

$\{P_1\}\{P_2\}\ e\ \{Q_1\}\{Q_2\} \stackrel{\text{def}}{=} \{P_1\}\ e\ \{Q_1\} \wedge \{P_2\}\ e\ \{Q_2\}$

Resource Sharing via Invariants

$$\{\, (x,y) \mapsto |0\rangle|\phi_0\rangle * [y] \,\}\ C_0X(x,y) \,||\, C_1H(x,y)\ \{\, (x,y) \mapsto |0\rangle \otimes X|\phi_0\rangle * [y] \,\}$$

$$\{\, y \mapsto |\phi_0\rangle * [y] \,\}\ C_0X(x,y) \,||\, C_1H(x,y)\ \{\, y \mapsto X|\phi_0\rangle * [y] \,\}^{x \mapsto |0\rangle}$$

Share $x \mapsto |0\rangle$ via the **invariant**

$$\{\, y \mapsto |\phi_0\rangle \,\}\ C_0X(x,y)\ \{\, y \mapsto X|\phi_0\rangle \,\}^{x \mapsto |0\rangle}$$

$$\{\, [y] \,\}\ C_1X(x,y)\ \{\, [y] \,\}^{x \mapsto |0\rangle}$$

$$\{\, x \mapsto |0\rangle * y \mapsto |\phi_0\rangle \,\}\ C_0X(x,y)\ \{\, x \mapsto |0\rangle * y \mapsto X|\phi_0\rangle \,\}$$

$$\{\, x \mapsto |0\rangle * [y] \,\}\ C_1X(x,y)\ \{\, x \mapsto |0\rangle * [y] \,\}$$

$$\frac{e \text{ is atomic} \quad \{\, P * I \,\}\, e\, \{\, Q * I \,\}}{\{\, P \,\}\ e\ \{\, Q \,\}^{I}}$$

$$\frac{\{\, P \,\}\ e\ \{\, Q \,\}^{I * J}}{\{\, P * I \,\}\ e\ \{\, Q * I \,\}^{J}}$$

$$\frac{\{P\}\, e\, \{Q\}^{I} \quad \{P'\}\, e'\, \{Q'\}^{I}}{\{P * P'\}\ e \,||\, e'\ \{Q * Q'\}^{I}}$$

Anti-Frame Rule by Atomicity

$$\{\, x \mapsto |0\rangle * [y] \,\}\ C_1X(x,y)\ \{\, x \mapsto |0\rangle * [y] \,\}$$

$$\frac{e \text{ is atomic} \qquad P\colon \text{out } x \qquad Q\colon \text{precise} \qquad \forall |\psi\rangle.\ \{\, x \mapsto |\psi\rangle * [x] * P \,\}\ e\ \{\, x \mapsto |\psi\rangle * [x] * Q \,\}}{\{\, [x] * P \,\}\ e\ \{\, [x] * Q \,\}}$$

Cf. Frame rule

$$\frac{\{P\}\, e\, \{Q\}}{\{P * R\}\, e\, \{Q * R\}}$$

- Qubit token $[x]$ allows atomic temporary writes to x
  - e.g., $I(x)$, atomic $\{\, X(x); (\ldots x \text{ is unchanged} \ldots); X(x) \,\}$
- Other processes can freely access x with the points-to token $x \mapsto |\psi\rangle$
  - Technically, qubit tokens can be used for *dirty qubits*

Complete Proof for $C_0X(x,y) \mid\mid C_1H(x,y)$

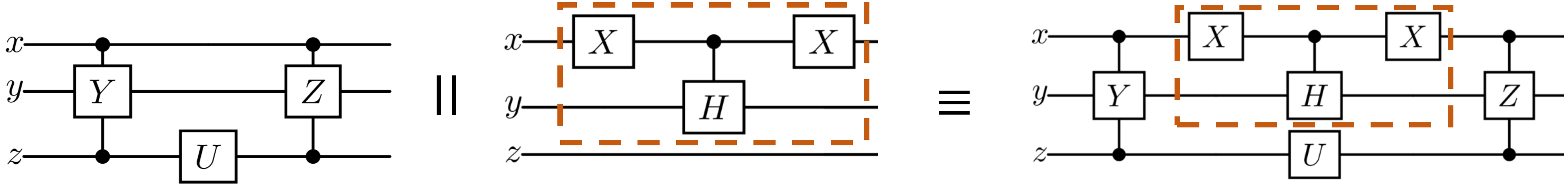
$$\{\, (x,y) \mapsto (\alpha|0\rangle|\phi_0\rangle + \beta|1\rangle|\phi_1\rangle) * [y] \,\}$$
$$\{\, x \mapsto |0\rangle * y \mapsto |\phi_0\rangle * [y] \,\}$$
$$\{\, x \mapsto |1\rangle * y \mapsto |\phi_1\rangle * [y] \,\}$$
$$\{\, y \mapsto |\phi_0\rangle * [y] \,\}^{x \mapsto |0\rangle}$$
$$\{\, y \mapsto |\phi_1\rangle * [y] \,\}^{x \mapsto |1\rangle}$$
$$C_0X(x,y) \ \mid\mid\ C_1H(x,y)$$
$$\{\, y \mapsto X|\phi_0\rangle * [y] \,\}^{x \mapsto |0\rangle}$$
$$\{\, y \mapsto H|\phi_1\rangle * [y] \,\}^{x \mapsto |1\rangle}$$
$$\{\, x \mapsto |0\rangle * y \mapsto X|\phi_0\rangle * [y] \,\}$$
$$\{\, x \mapsto |1\rangle * y \mapsto H|\phi_1\rangle * [y] \,\}$$
$$\{\, (x,y) \mapsto (\alpha|0\rangle \otimes X|\phi_0\rangle + \beta|1\rangle \otimes H|\phi_1\rangle) * [y] \,\}$$

$$\{\, y \mapsto |\phi_0\rangle \,\}^{x \mapsto |0\rangle} \ \{\, [y] \,\}^{x \mapsto |1\rangle}$$
$$\{\, y \mapsto |\phi_0\rangle * x \mapsto |0\rangle \,\}\{\, [y] * x \mapsto |1\rangle \,\}$$
$$C_0X(x,y)$$
$$\{\, y \mapsto X|\phi_0\rangle * x \mapsto |0\rangle \,\}\{\, [y] * x \mapsto |1\rangle \,\}$$
$$\{\, y \mapsto X|\phi_0\rangle \,\}^{x \mapsto |0\rangle} \ \{\, [y] \,\}^{x \mapsto |1\rangle}$$

$$\{\, [y] \,\}^{x \mapsto |0\rangle} \ \{\, y \mapsto |\phi_1\rangle \,\}^{x \mapsto |1\rangle}$$
$$\{\, [y] \,\} \ \{\, y \mapsto |\phi_1\rangle * x \mapsto |1\rangle \,\}$$
$$C_1H(x,y)$$
$$\{\, [y] * x \mapsto |0\rangle \,\} \ \{\, y \mapsto H|\phi_1\rangle * x \mapsto |1\rangle \,\}$$
$$\{\, [y] \,\}^{x \mapsto |0\rangle} \ \{\, y \mapsto H|\phi_1\rangle \,\}^{x \mapsto |1\rangle}$$

$$\{P_1\}\,\{P_2\}\; e\; \{Q_1\}\,\{Q_2\} \ \stackrel{\text{def}}{=}\ \{P_1\}\, e\, \{Q_1\} \wedge \{P_2\}\, e\, \{Q_2\}$$

More Complex Example



$$\{(x, y, z) \mapsto (\alpha|0\rangle|\psi_{yz}\rangle + \beta|1\rangle|\phi_{yz}\rangle) * [y] * [z] * \cdots\}$$

Invariant
$x \mapsto |0\rangle * [y] * [z]$ || Invariant
$x \mapsto |0\rangle * (y, z) \mapsto |\psi_{yz}\rangle$

$CCY(x, z, y); U(z); CCZ(x, z, y)$ || atomic $\{ X(x); CH(x, y); X(x) \}$

x is updated only **temporarily**

$$\{ (x, y, z) \mapsto (\alpha|0\rangle \otimes H_y U_z|\psi_{yz}\rangle + \beta|1\rangle \otimes CY_{zy} U_z CZ_{zy}|\phi_{yz}\rangle) * \cdots \}$$

Another Fun Thing: Commuting Matrices

We can verify parallelization of arbitrary commuting matrices

- Since commutative matrices are simultaneously diagonalizable

$\{ x \mapsto (\alpha|0\rangle + \beta|1\rangle) \}$

$\{ x \mapsto |0\rangle \} \{ x \mapsto |1\rangle \}$

$\{ () \mapsto 1 \}^{x \mapsto |0\rangle} \{ () \mapsto 1 \}^{x \mapsto |1\rangle}$

$R_{\theta_1}(x) \,||\, R_{\theta_2}(x)$

$\{ () \mapsto 1 \}^{x \mapsto |0\rangle} \underline{\{ () \mapsto e^{i(\theta_1+\theta_2)} \}}^{x \mapsto |1\rangle}$

$\{ x \mapsto |0\rangle \} \{ x \mapsto e^{i(\theta_1+\theta_2)}|1\rangle \}$

$\{ x \mapsto (\alpha|0\rangle + \beta e^{i(\theta_1+\theta_2)}|1\rangle) \}$

$R_{\theta_1}(x)$ and $R_{\theta_2}(x)$ have the same eigenvectors $\{|0\rangle, |1\rangle\}$
$\Rightarrow$ Quantum case analysis by $|0\rangle, |1\rangle$

Global phases can be tracked with empty-qubit points-to tokens

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- **Extension to Probabilistic Reasoning & Conclusion**

Extension to Probabilistic Reasoning

- Want to support quantum measurements!
- Challenge: Precise reasoning about **probabilistic** behavior
  - **Density matrix**, probabilistic distribution modulo equalities
    - e.g., $\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \; = \; \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \; = \; \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Our idea: Refine **Demonic Outcome Logic** [Zilberstein+ POPL'25] & its CSL variant [Zilberstein+ arXiv]
  - Key mechanism: **Probabilistic combination** $P +_p Q$
    - Solves the limitations of the existing quantum SL [Le+ POPL'22]
  - Model: **Convex PCM** (new!), a hybrid of convex space & PCM

Two-Layer Logic

Layer1: Vector-based Quantum Separation Logic (I talked today)

- Quantum points-to token (as vector) $\overline{x} \mapsto |\psi\rangle$ and qubit token $[x]$
- Sensitive to global phase because of linear combination of Hoare triples

“densify”

Layer2: Probabilistic Quantum Separation Logic (New)

- Quantum points-to token (as *density operator*) $\overline{x} \mapsto \rho$
- Probabilistic combination $P +_p Q$
- Support quantum measurements
- Insensitive global phase

Conclusion

- We proposed a concurrent quantum separation logic for modular verification of fine-grained parallelism
- Our logic supports shared quantum resources via invariants, the linear combination rule, and the anti-frame rule by atomicity
- Future work
  - More powerful concurrency reasoning
  - Automated optimization of quantum programs & its verification

Our Target Language

$e ::= x \mid l \mid n \mid () \mid op(\bar{e})$

$\mid$ qalloc (qubit allocation)

$\mid$ qfree e (qubit deallocation)

$\mid U(\bar{e})$ (quantum gate)

$\mid$ meas(e) (qubit measurement)

Quantum

$\mid e \mid\mid e'$ (parallel execution)

$\mid$ atomic $\{ e \}$ (atomic block)

Concurrency

$\mid\ !e \mid e \leftarrow e' \mid \cdots$ (heap)

$\mid$ if e $\{ e' \}$ else $\{ e'' \}$ $\mid$ while e $\{ e' \}$ $\mid \cdots$

Overview of Our Logic

Invariant

$$\{P\}\, e\, \{v.\, Q_v\}^{I}$$

$$\{P\}\, e\, \{v.\, Q_v\} \triangleq \{P\}\, e\, \{v.\, Q_v\}^{\mathrm{emp}}$$

$$P ::= \top \mid \bot \mid \neg P \mid P \land Q \mid P \lor Q$$
$$\mid P \to Q \mid \forall a.\, P_a \mid \exists a.\, P_a$$
$$\mid \mathrm{emp} \mid P * Q \mid P -\!\!* \, Q \quad \text{(SL connectives)}$$
$$\mid l \mapsto v \mid \bar{x} \mapsto |\psi\rangle \mid [x]$$

$$\{\, \mathrm{emp}\,\}\ \mathrm{qalloc}\ \{\, x.\ x \mapsto |0\rangle * [x]\,\} \qquad \{\, x \mapsto |0\rangle * [x]\,\}\ \mathrm{qfree}\, x\ \{\, \mathrm{emp}\,\}$$

$$\{\, \bar{x} \mapsto |\psi\rangle\,\}\ U(\bar{x})\ \{\, \bar{x} \mapsto U|\psi\rangle\,\}$$

… and more interesting rules!

- ***Quantum points-to token*** $\bar{x} \mapsto |\psi\rangle$: the state vector of $\bar{x}$ is $|\psi\rangle$
- Separation $*$ means disentangled qubit states:
  $$\bar{x} \mapsto |\psi\rangle * \bar{y} \mapsto |\phi\rangle \equiv (\bar{x}, \bar{y}) \mapsto |\psi\rangle \otimes |\phi\rangle$$
- ***Qubit token*** $[x]$ (new!): Qubit x is alive, but its state is unknown

Linear Combination Rule

- This idea can be formalized as **linear combination of Hoare triples**

$$\frac{\{\, \bar{x} \mapsto |\psi\rangle * P \,\}\ e\ \{\, \bar{x} \mapsto |\phi\rangle * Q \,\}^{I} \quad \{\, \bar{x} \mapsto |\psi'\rangle * P \,\}\ e\ \{\, \bar{x} \mapsto |\phi'\rangle * Q \,\}^{I}}{\{\, \bar{x} \mapsto (\alpha|\psi\rangle + \beta|\psi'\rangle) \ * P \,\}\ e\ \{\, \bar{x} \mapsto (\alpha|\phi\rangle + \beta|\phi'\rangle) \ * Q \,\}^{I}}$$

  - With the side condition Q, I: precise
    - Precise assertions represent a unique (or no) resource
      - e.g., $\mathrm{emp}, \bot, l \mapsto v, x \mapsto |\psi\rangle, l \mapsto v * x \mapsto |\psi\rangle, \ldots$
    - If not I: precise, the angelic branching on I makes the rule unsound

Now Our Subgoals:

$$\{\, (x,y) \mapsto |0\rangle|\phi_0\rangle * [y] \,\}\ C_0X(x,y) \,||\, C_1H(x,y)\ \{\, (x,y) \mapsto |0\rangle \otimes X|\phi_0\rangle * [y] \,\}$$

$$\{\, (x,y) \mapsto |1\rangle|\phi_1\rangle * [y] \,\}\ C_0X(x,y) \,||\, C_1H(x,y)\ \{\, (x,y) \mapsto |1\rangle \otimes H|\phi_1\rangle * [y] \,\}$$

Teaser of Our Probabilistic Quantum SL

- On probabilistic combinations

$$P +_p Q \;\equiv\; Q +_{1-p} P \qquad \big(P +_p Q\big) +_q R \;\equiv\; P +_{pq} \big(Q +_{\frac{(1-p)q}{1-pq}} R\big)$$

$$P \vdash P +_p P \qquad P\text{: convex} \;\stackrel{\text{def}}{\equiv}\; \forall p.\; P +_p P \equiv P$$

Convex hull modality $\triangle P \;\stackrel{\text{def}}{\equiv}\; \exists\, \bar{p} \in (0,1)^* \text{ s.t.} \sum \bar{p} = 1.\, \sum_i p_i P$

$$P \vdash \triangle P \qquad \triangle\triangle P \equiv \triangle P \qquad \triangle P\text{: convex} \qquad \triangle\big(P +_p Q\big) \equiv \triangle P +_p \triangle Q$$

$$\big(P +_p Q\big) * R \;\equiv\; P * R +_p Q * R \quad \text{if } R\text{: precise}$$

$$\{\,\text{emp}\,\}\;\, v \oplus_p v' \;\,\{\,\langle v\rangle +_p \langle v'\rangle\,\} \qquad \{\,\text{emp}\,\}\;\, \text{ndint} \;\,\{\,\triangle\,(\exists n.\, \langle n\rangle)\,\}$$

- Quantum

$$\bar{x} \mapsto \rho +_p \bar{x} \mapsto \rho' \;\equiv\; \bar{x} \mapsto (p\rho + (1-p)\rho')$$

$$\{\,x \mapsto \rho\,\}\;\, \text{meas}(x) \;\left\{ \langle 0\rangle * x \mapsto \tfrac{1}{p} Pr_0 \rho Pr_0 +_p \langle 1\rangle * x \mapsto \tfrac{1}{1-p} Pr_1 \rho Pr_1 \right\}$$

where $p = \text{tr}(Pr_0 \rho)$

Basics of Quantum Computing

- State for a ***qubit (quantum bit)*** = *2D vector* $|\psi\rangle \in \mathbb{C}^2$

  Superposition $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \quad \alpha, \beta \in \mathbb{C} \quad |\alpha|^2 + |\beta|^2 = 1$

- State for ***n*** qubits = Vector of *tensor product* space $\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2 \cong \mathbb{C}^{2^n}$
  - Composite of $|\psi\rangle$ and $|\phi\rangle$ = *Tensor product* $|\psi\rangle \otimes |\phi\rangle = |\psi\rangle|\phi\rangle = |\psi\phi\rangle$

- ***Quantum gate*** = *Unitary matrix* $U : \mathcal{H} \to \mathcal{H}$

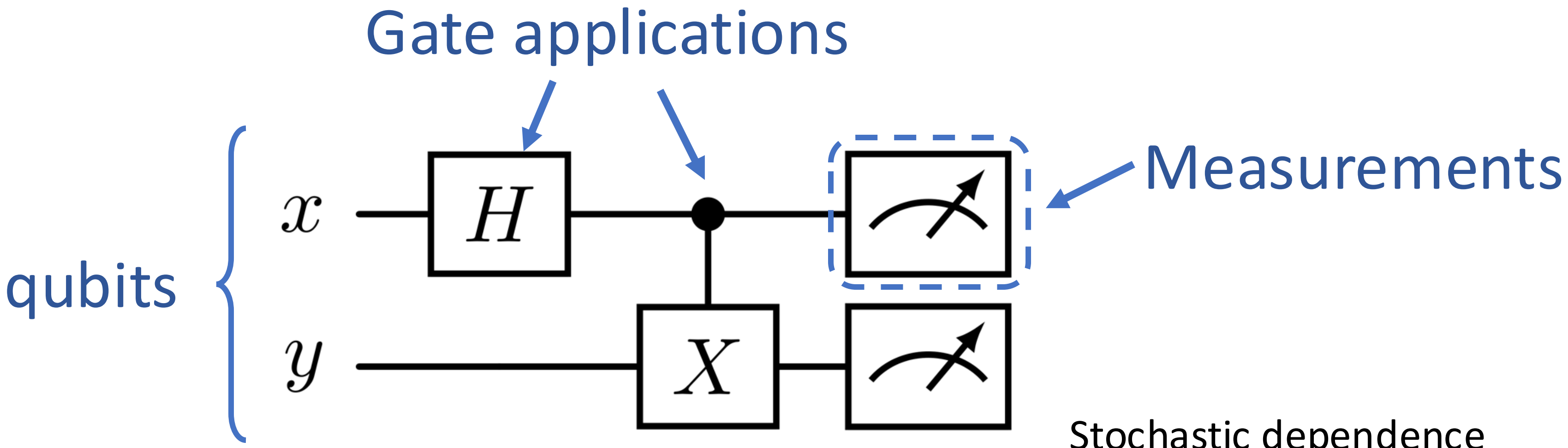
  e.g., $H|b\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^b|1\rangle\right) \quad CX|b\rangle|c\rangle = |b\rangle|b \text{ xor } c\rangle \quad b, c \in \{0,1\}$

  Hadamard — Controlled X

- ***Measurement*** = *Probabilistic branching & convergence*

$$\alpha|0\rangle + \beta|1\rangle \to \begin{cases} |0\rangle & (w.p. \ \ |\alpha|^2) \\ |1\rangle & (w.p. \ \ |\beta|^2) \end{cases}$$

Quantum Program (Circuit)



Stochastic dependence

$$x, y \mapsto |00\rangle \to |+\rangle|0\rangle \to \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \to \begin{cases} |00\rangle & (w.p.\ 1/2) \\ |11\rangle & (w.p.\ 1/2) \end{cases}$$

$$|\pm\rangle \overset{\text{def}}{=} \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

Entangled state

x and y are entangled $\iff$ $x, y \mapsto |\psi\rangle$ such that $\forall |\phi\rangle, |\phi'\rangle.\ |\psi\rangle \neq |\phi\rangle \otimes |\phi'\rangle$